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THE TRANSITION TO INSTABILITY IN WEAKLY NON-UNIFORM FLOWS WITHOUT DISSIPATION†

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A criterion which enables one to determine in what cases breakdown of the local stability conditions leads to instability of weakly non-uniform flow without dissipation is obtained. Copyright © 1996 Elsevier Science Ltd.

We will consider the stability of flows which depend slowly on one spatial variable x, i.e. flows in which all the quantities depend on $X = x/L$, where the characteristic scale of length L is fairly large (this is in fact necessary in order that the scale L should be much greater than the characteristic lengths of the perturbations, which will be considered below). If we "freeze" the flow parameters, making them constant and equal to the parameters of the flow for a certain value X, we can consider perturbations of the form $\exp(ikx - i\omega t)$ and obtain a dispersion relation in which X will occur as a parameter

$$\Phi(\omega, k, X) = 0 \tag{1}$$

From (1) we can obtain the frequency ω, which is a multivalued analytic function of k and X. We will consider the case when, for real X and k, real or complex conjugate values of ω satisfy (1). If Eq. (1) is a polynomial in k and ω, the case considered corresponds to coefficients of this polynomial that are real. This is a typical situation for flows without dissipation (see, for example, [1]). In the case considered, the frozen flow will be locally stable if all the ω are real for an arbitrarily chosen X.

Suppose the flow depends on a certain parameter R, such that a transition to instability occurs when R becomes R_*. We will assume that a transition to local instability occurs as follows. When R increases, a small interval (X_1, X_2) appears on the X axis, for points of which there is a small range of real values of k with complex values of ω. When $X \subset (X_1, X_2)$ reclosure of the branches of the multivalued function $\omega(k)$ occurs on the graph of the real values of ω as a function of real values of k (henceforth we will assume that reclosure of two branches occurs) and a section is formed on the k axis, where the number of real values of the function $\omega(k)$ is less than previously (Figs 1 and 2). Below we will consider this transition and investigate in what cases it leads to instability of the whole flow, and in what cases it does not.

When the local instability described above occurs two versions are possible: the local instability is absolute or convective [2, 3]. The criterion which distinguish absolute instability and convective instability in a non-dissipative medium can be formulated as follows. If simultaneously with the occurrence of a range on the k axis with complex values of ω there is no interval on the ω axis with complex k (Fig. 1), the instability is absolute. If the occurrence of a range on the k axis with complex values of ω is accompanied by the occurrence of a range on the ω axis with complex values of k (the interval (ω_1, ω_2) in Fig. 2), the instability is convective [3]. We emphasize that the ideas of absolute and convective instability relate to flow with frozen parameters.

For the local transition to instability considered when $R = R_*$ and the unique value $X = X_*$, there is a point $\omega = \omega_*$, $k = k_*$ in the k, ω plane at which the branches of the $\omega(k)$ graph intersect. For a slight supercriticality, i.e. for small $R - R_* > 0$, the dispersion relation (1) can be expanded in series in the neighbourhood of this point and can be written in the following form, for the principal terms

$$(\omega' - Uk')^2 = Ak'^2 - \alpha + \beta X'^2$$
$$\omega' = \omega - \omega_*, \quad k' = k - k_*, \quad X' = X - X_* \tag{2}$$

Here $U, A > 0$, $\beta > 0$ are constants, the value of α depends on R, and $\alpha(R_*) = 0$, $\alpha(R) > 0$ when $R > R_*$. Obviously when $\alpha - \beta X'^2 > 0$ local instability occurs.

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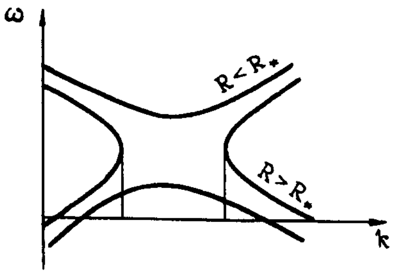

Fig. 1.

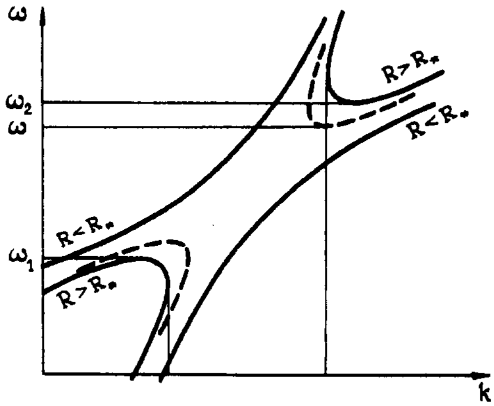

Fig. 2.

We will first consider the case when the local instability that occurs when R reaches the value of R_* is absolute. Then, for values of R close to R_*, there is a point X at which the following relations are simultaneously satisfied

$$\partial\omega'/\partial k' = 0, \qquad \partial\omega'/\partial X' = 0 \tag{3}$$

As will be shown below, in this case instability occurs due to the fact that Eqs (3) are satisfied and due to the formation of an eigenfunction at two close turning points. Such instability was considered previously in [4–6].

In the case of absolute instability, the coefficients of (2) satisfy the condition $U^2 < A$. When the latter inequality is satisfied, and also when $\alpha - \beta X'^2 > 0$, we obtain from the first equation of (3) and Eq. (2)

$$k' = k_s'(X') \equiv -\frac{iU\sqrt{\alpha-\beta X'^2}}{\sqrt{A(A-U^2)}}, \quad \omega' = \omega_s'(X') \equiv i\sqrt{\frac{(\alpha-\beta X'^2)(A-U^2)}{A}} \tag{4}$$

In (4) we have chosen the branch of ω' for which $\operatorname{Im}\omega' > 0$ at the saddle point $k_s'(X')$.

The value of $X' = 0$ satisfies the second equation of (3). Hence, values $X' = 0$, $\omega' = \omega_s'(0)$, $k' = k_s'(0)$ satisfy both equations of (3). At the point $X' = 0$ the analytic function $\omega_s'(X')$ has a derivative equal to zero, while the harmonic function $\operatorname{Im}\omega_s'(X')$ has a saddle point. As follows from (4), $\operatorname{Im}\omega_s'(X') < \operatorname{Im}\omega_s'(0)$ for points on the real X' axis and $\operatorname{Im}\omega_s'(X') > \operatorname{Im}\omega_s'(0)$ for points on the imaginary X' axis. Under these conditions for perturbations of the non-uniform flow considered (which depend on x/L) there are natural frequencies ω' close to $\omega_s'(0)$ (so that $\omega' - \omega_s'(0) = O(1/L)$). The eigenfunction is related only to the two branches of $k(\omega, X)$ and the two turning points in the X plane, which lie on the real X axis close to the point $X = X_*$. These branches of $k_1^1(\omega, X)$ and $k_2^1(\omega, X)$ correspond to waves propagating in different directions and are described by Eq. (2). At the turning points these values of k^1 are identical, while the turning points themselves when $R = R_*$ merge into the saddle point $X = X_*$. The construction of the eigenfunction is described in [6] for a somewhat different dispersion relation, but it can easily be adapted to the case considered here.

It can be shown that for pure imaginary ω', somewhat less in modulus than $|\omega_s'(0)|$, there are two turning points $\pm X_t'$, which lie on the real X' axis. This can be seen from the second equation of (4), if we regard it as an equation in X'. It follows from (2) that the difference in values $k_2' - k_1'$ between turning points is real, and it becomes complex outside the section $[-X_t', X_t']$. For large values of L this enables the eigenfunction to be constructed using the standard WKB method [7] which, in the section $[-X_t^1, X_t^1]$ is the sum of two terms proportional to $\exp(i\int k_1(\omega, x)dx)$ and $\exp(i\int k_2(\omega, x)dx)$, respectively. This eigenfunction decays outside the section $[-X_t', X_t']$. The natural frequencies ω' must satisfy the equation [8]

$$\int_{-x_t}^{x_t} [k_1(\omega', x) - k_2(\omega', x)]dx = \pi\left(n + \frac{1}{2}\right)$$

where n is an integer and $x_t = LX_t$. The value of ω' which satisfies this equation can be chosen to be closer to $\omega_s'(0)$ the larger the value of L.

We will now consider the case when, for $R > R_*$ a local instability occurs which is convective (Fig. 2). Here, in relation (2) $U^2 > A$ and, which is most important, branching of the function $k(\omega, X)$ occurs for waves travelling in the same direction. We will assume, as in Fig. 2, that they propagate to the right. In this case, for real ω, such that $\omega_1 < \omega < \omega_2$ (where (ω_1, ω_2) is the range of values to which complex k corresponds when $X' = 0$), there are two turning points corresponding to real values of $X' = \pm X_t'$, for which in Fig. 2 the straight line ω = const touches the dispersion curve $\omega(k)$. For real ω and X the values of k_1 and k_2 take complex conjugate values between these turning points. Then, one of the WKB waves $\exp(i\int kdx - i\omega t)$ is amplified ($\mathrm{Im}\, k_1' < 0$) during its motion (as x increases), while the other is attenuated ($\mathrm{Im}\, k_2' > 0$) due to the factor $\exp(i\int kdx)$.

At turning points the usual conversion of the waves with real k into waves with complex k (the Stokes phenomenon) occurs, as for second-order equations. If waves corresponding to real $k_1'(\omega, X)$ and $k_2'(\omega, X)$ arrive at the point $X' = -X_t'$ from the left, then, in general, there will also be both waves corresponding to complex $k_1'(\omega, X)$ and $k_2'(\omega, X)$ from the right of this point. In order that there should be no amplified wave corresponding to $k_1'(\omega, X)$ more to the right of the turning point, there has to be a strict relationship between the amplitudes and phases of the waves arriving at this point. This case will be regarded as exceptional and will not be considered. Then, of the pair of waves arising at the point $X' = -X_t$, one arrives at the points X_t amplified and the second one arrives attenuated. If the length of the section of the x axis between turning points is sufficiently long, so that $\mathrm{Im}(\int kdx) \gg 1$, where the integration is carried out between turning points, the attenuated wave at the point X_t can be neglected. The amplified wave is converted at the point X_t into two waves with the same amplitudes, which correspond to real $k_1'(\omega, X)$ and $k_2'(\omega, X)$, and both these waves propagate to the right. Hence, the section $[-X_t', X_t']$ between turning points act as a wave amplifier. This corresponds to the representation of the mechanism of convective instability as an amplifier [3]. This same amplification mechanism will obviously act not only for real ω but also for ω with values of $\mathrm{Im}\, \omega > 0$ that are not too large.

However, the presence of an amplifier is insufficient for instability to occur. Feedback is also necessary. It is necessary for the amplified wave to return once more to the input of the amplifier. Since the local instability can only occur on one section of the X axis, according to the above proposition, the signal can only return to the amplified input due to reflections of the waves corresponding to $k(\omega)$ branches that are real for real ω. If we draw the graph of real $k(\omega, X)$ for specified real ω, then in order to form a perturbation that increases with time it is necessary that at least one of the branches $k_1(X)$ or $k_2(X)$ should represent part of the closed curve for $R < R_*$ (the dashed curve in Fig. 3). In this case when $R < R_*$ there are eigenfunctions corresponding to real values of ω [9]. This eigenfunction, for the case shown in Fig. 3, consists of waves travelling in different directions, corresponding to $k_2(\omega, X)$ and $k_3(\omega, X)$, which are converted into one another at turning points of rotation (at turning points the tangents to the $k(X)$ graph are vertical). When $R > R_*$ the amplifier described above is inserted into this chain of waves, leading to an increase in the perturbations with time. This behaviour of the perturbations occurred in the specific problem described in [10].

If when $R < R_*$ there is no chain of waves with real k (i.e. there are no eigenfunctions corresponding to real ω), then for small $R - R_* > 0$ no instability occurs since there is no effective feedback.

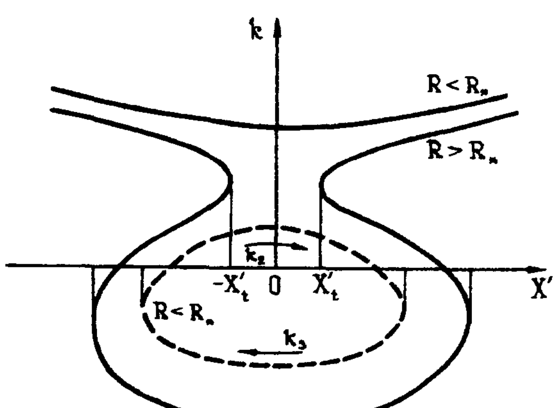

Fig. 3.

Note that, generally speaking, there can also be turning points lying in the complex X plane. However, the reflection coefficients of these turning points are exceptionally small [7]. For small $R - R_*$ and correspondingly small X_t', amplification of the waves in the section $[-X_t', X_t']$ will also be small and insufficient to give rise to a growing perturbation. It can occur for large values of $R - R_*$ when amplification of the waves turns out to be large and a growing chain of waves is formed, despite the smallness of the reflection coefficients of the complex turning points. We will not consider this case here.

Hence, we have shown that when the dispersion relation is such that real or complex conjugate values of ω correspond to real k and X, the occurrence of a region of local instability on the X axis leads to the occurrence of growing perturbations of non-uniform flow in two cases: (a) when the local instability that occurs is absolute, and (b) when the local instability that occurs for $R > R_*$ is convective, but when $R < R_*$ there is a chain of interconverting waves corresponding to real ω. In the remaining cases, for sufficiently small $R - R_*$, the flow remains stable in the linear approximation.

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